Derived categories

- What are they?
- Geometrically?
- N.C. algebra?
- Physics?

R a ring $\rightsquigarrow$ R-mod
modules over R
"Abelian category"

Affine scheme = rings $\rightsquigarrow$ R-mods
affine variety Spec R

Projective varieties
(or any) are locally affine, Spec R

"coherent sheaf" $\leftarrow$ locally a R-module.

vector bundle $\leftarrow$ locally a free module.

Variety $X \rightsquigarrow$ category $\mathrm{Coh}(X)$
(abelian)

$$\begin{tikzcd} & \text{algebra} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \text{geometry} \arrow[rr, leftrightarrow] & & \text{cat. thy} \end{tikzcd}$$

Abelian cat $\rightsquigarrow$ "derived category" natural setting for Ext, Tor, H^0, ...
$\cup$
chain complexes

variety
$X \rightsquigarrow D^b(X)$ or $D^b(\mathrm{Coh}\, X)$
"categories"

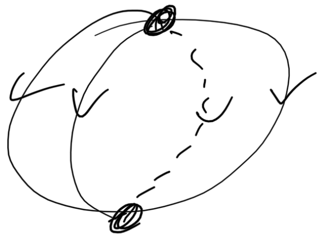
$pt = \mathrm{Spec}\, \mathbb{C} \rightsquigarrow \mathrm{Vect}_{\mathbb{C}}$ (or $D^b(\mathrm{Vect}_{\mathbb{C}})$).

$\mathbb{P}^1 \longrightarrow D^b(\mathbb{P}^1)$ turns out has a "semi-orthogonal decomposition" (SOD)

$$D^b(\mathbb{P}^1) = \langle \underbrace{\mathrm{Vect}}_{D^b(pt)}, \underbrace{\mathrm{Vect}}_{D^b(pt)} \rangle$$

"full excep. collection"

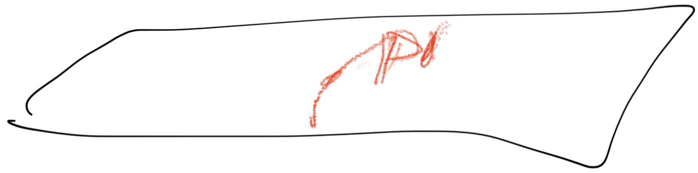
"Beilinson Thm"



$$\overset{2}{\mathbb{C}} \longrightarrow \overset{1}{\mathcal{O}} \longrightarrow \overset{0}{\mathbb{C}}$$

$$D^b(\mathbb{P}^n) = \langle \underbrace{\mathrm{Vect}, \ldots, \mathrm{Vect}}^{n+1 \text{ times}} \rangle$$

"Blow-ups"



$$\mathbb{A}^2 \ni (0,0) \quad \{\text{lines through } (0,0)\} = \mathbb{P}^1$$

$$\exists \text{ space } Bl_{(0,0)}\mathbb{P}^1 \xrightarrow{\pi} \mathbb{A}^2$$

π iso away from $(0,0)$

$$\pi^{-1}(0,0) = \mathbb{P}^1$$

e.g. of a birational transformation.

Fact

$$D^b(Bl_{(0,0)}\mathbb{A}^2) = \langle D^b(\mathbb{A}^2), D^b(pt) \rangle$$

$$D^b(Bl_0 \mathbb{A}^n) = \langle D^b(\mathbb{A}^n), \underbrace{D^b(pt), \ldots, D^b(pt)}_{n-1} \rangle$$

Birational equivalences are related to decomposing $D^b(X)$?

In 2d, surfaces have unique minimal birational models.

In 3d,

$$\begin{tikzcd} & \tilde{X} \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed, leftrightarrow, "\text{"flop"}"'] & & X' \end{tikzcd}$$

get non-unique minimal models

Conj: $D^b(X) \cong D^b(X')$

verified in many e.g.s in dim 3.

Algebra: $R \rightsquigarrow R\text{-mods} \; (\rightsquigarrow D^b(R))$

$A \rightsquigarrow A\text{-mods} \; (\rightsquigarrow D^b(A))$

n.c. alg geometry??

A: $\bullet \overset{x}{\underset{y}{\rightrightarrows}} \bullet$ "path algebra of this quiver"

"quiver"

$$D^b(\mathbb{P}^1) \cong D^b(A)$$

Physics: B-branes!